Lecture 4 : General Logarithms and Exponentials.

For $a > 0$ and x any real number, we define

$$\boxed{a^x = e^{x \ln a}, \quad a > 0.}$$

The function a^x is called the exponential function with base a.
Note that $\ln(a^x) = x \ln a$ is true for all real numbers x and all $a > 0$. (We saw this before for x a rational number).
Note: We have no definition for a^x when $a < 0$, when x is irrational.

For example $2^{\sqrt{2}} = e^{\sqrt{2}\ln 2}$, $2^{-\sqrt{2}}$, $(-2)^{\sqrt{2}}$(no definition).

Algebraic rules

The following **Laws of Exponent** follow from the laws of exponents for the natural exponential function.

$$\boxed{a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x}$$

Proof $a^{x+y} = e^{(x+y)\ln a} = e^{x \ln a + y \ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y$. etc...

Example Simplify $\frac{(a^x)^2 a^{x^2+1}}{a^2}$.

Differentiation

The following **differentiation rules** also follow from the rules of differentiation for the natural exponential.

$$\boxed{\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = a^x \ln a \qquad \frac{d}{dx}(a^{g(x)}) = \frac{d}{dx} e^{g(x)\ln a} = g'(x) a^{g(x)} \ln a}$$

Example Differentiate the following function:

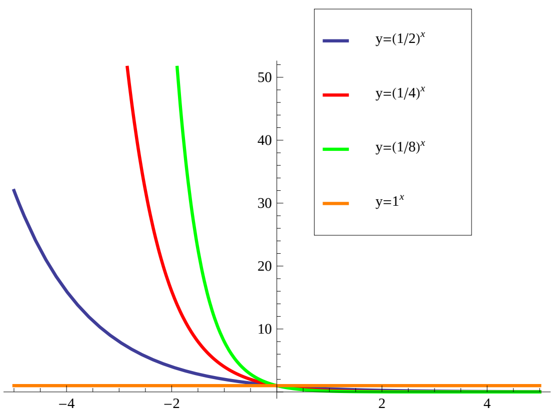
$$f(x) = (1000)2^{x^2+1}.$$

Graphs of Exponential functions. Case 1: $0 < a < 1$

- y-intercept: The y-intercept is given by $y = a^0 = e^{0 \ln a} = e^0 = 1$.
- x-intercept: The values of $a^x = e^{x \ln a}$ are always positive and there is no x intercept.

- Slope: If $0 < a < 1$, the graph of $y = a^x$ has a negative slope and is always decreasing, $\frac{d}{dx}(a^x) = a^x \ln a < 0$. In this case a smaller value of a gives a steeper curve.
- The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x(\ln a)^2 > 0$.
- As $x \to \infty$, $x \ln a$ approaches $-\infty$, since $\ln a < 0$ and therefore $a^x = e^{x \ln a} \to 0$.
- As $x \to -\infty$, $x \ln a$ approaches ∞, since both x and $\ln a$ are less than 0. Therefore $a^x = e^{x \ln a} \to \infty$.

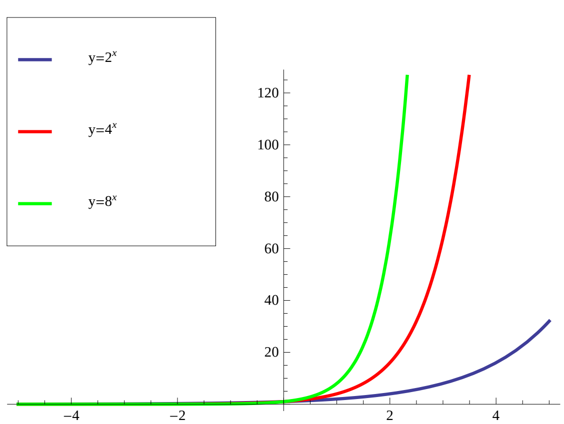
$$\boxed{\text{For } 0 < a < 1, \quad \lim_{x\to\infty} a^x = 0, \quad \lim_{x\to-\infty} a^x = \infty}.$$



Graphs of Exponential functions. Case 2: $a > 1$

- y-intercept: The y-intercept is given by $y = a^0 = e^{0 \ln a} = e^0 = 1$.
- x-intercept: The values of $a^x = e^{x \ln a}$ are always positive and there is no x intercept.
- If $a > 1$, the graph of $y = a^x$ has a positive slope and is always increasing, $\frac{d}{dx}(a^x) = a^x \ln a > 0$.
- The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x(\ln a)^2 > 0$.
- In this case a larger value of a gives a steeper curve.
- As $x \to \infty$, $x \ln a$ approaches ∞, since $\ln a > 0$ and therefore $a^x = e^{x \ln a} \to \infty$
- As $x \to -\infty$, $x \ln a$ approaches $-\infty$, since $x < 0$ and $\ln a > 0$. Therefore $a^x = e^{x \ln a} \to 0$.

$$\boxed{\text{For } a > 1, \quad \lim_{x\to\infty} a^x = \infty, \quad \lim_{x\to-\infty} a^x = 0}.$$



Functions of the form $(f(x))^{g(x)}$.

Derivatives We now have 4 different types of functions involving bases and powers. So far we have dealt with the first three types:
If a and b are constants and $g(x) > 0$ and $f(x)$ and $g(x)$ are both differentiable functions.

$$\frac{d}{dx}a^b = 0, \qquad \frac{d}{dx}(f(x))^b = b(f(x))^{b-1}f'(x), \qquad \frac{d}{dx}a^{g(x)} = g'(x)a^{g(x)}\ln a, \qquad \frac{d}{dx}(f(x))^{g(x)}$$

For $\frac{d}{dx}(f(x))^{g(x)}$, we use **logarithmic differentiation or write the function as** $(f(x))^{g(x)} = e^{g(x)\ln(f(x))}$ **and use the chain rule**.

Example Differentiate x^{2x^2}, $x > 0$.

Limits

To calculate limits of functions of this type it may help write the function as $(f(x))^{g(x)} = e^{g(x)\ln(f(x))}$.

Example What is $\lim_{x\to\infty} x^{-x}$

General Logarithmic functions

Since $f(x) = a^x$ is a monotonic function whenever $a \neq 1$, it has an inverse which we denote by $f^{-1}(x) = \log_a x$. We get the following from the properties of inverse functions:

$$f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x$$

$$\boxed{\log_a(x) = y \quad \text{if and only if} \quad a^y = x}$$

$$f(f^{-1}(x)) = x \qquad f^{-1}(f(x)) = x$$

$$\boxed{a^{\log_a(x)} = x \qquad \log_a(a^x) = x.}$$

Converting to the natural logarithm

It is not difficult to show that $\log_a x$ has similar properties to $\ln x = \log_e x$. This follows from the **Change of Base Formula** which shows that The function $\log_a x$ is a constant multiple of $\ln x$.

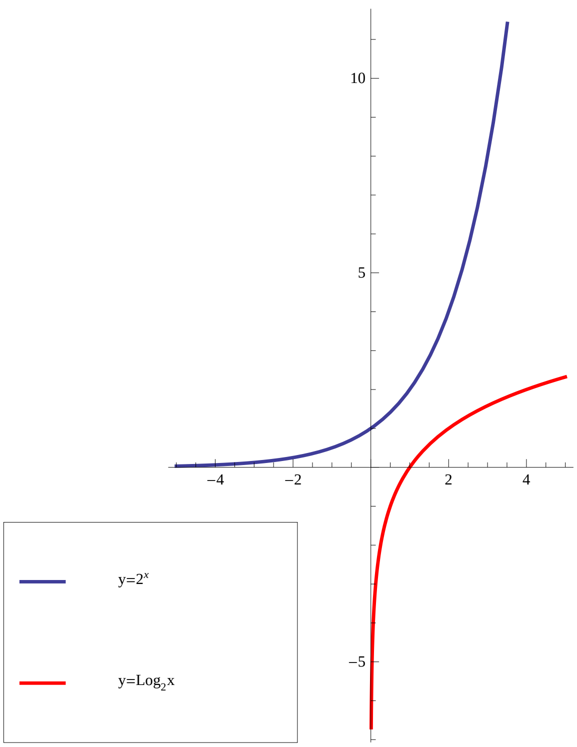
$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

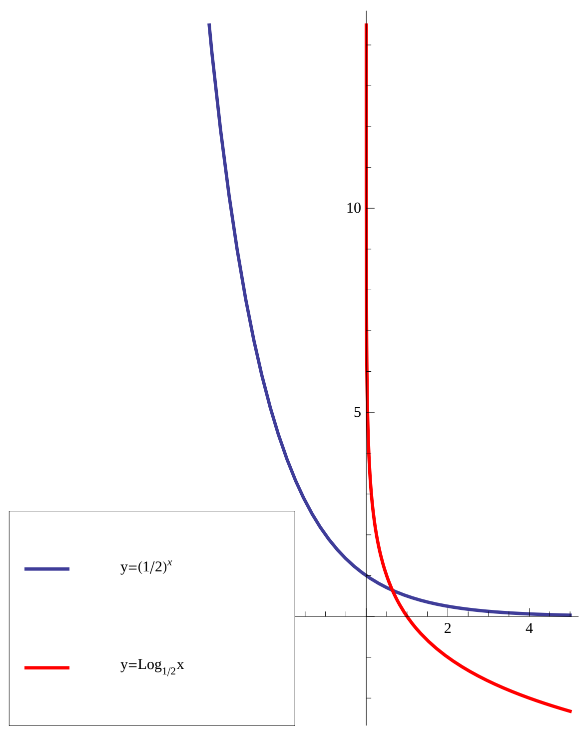
The algebraic properties of the natural logarithm thus extend to general logarithms, by the change of base formula.

$$\log_a 1 = 0, \qquad \log_a(xy) = \log_a(x) + \log_a(y), \qquad \log_a(x^r) = r\log_a(x).$$

for any positive number $a \neq 1$. In fact for most calculations (especially limits, derivatives and integrals) it is advisable to convert $\log_a x$ to natural logarithms. The most commonly used logarithm functions are $\log_{10} x$ and $\ln x = \log_e x$.

Since $\log_a x$ is the inverse function of a^x, it is easy to derive the properties of its graph from the graph $y = a^x$, or alternatively, from the change of base formula $\log_a x = \frac{\ln x}{\ln a}$.





Basic Application

Example Express as a single number $\log_5 25 - \log_5 \sqrt{5}$

Using the change of base formula for Derivatives

From the above change of base formula for $\log_a x$, we can easily derive the following **differentiation formulas**:

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\log_a g(x)) = \frac{g'(x)}{g(x) \ln a}.$$

Example Find $\frac{d}{dx} \log_2(x \sin x)$.

A special limit and an approximation of e

We derive the following limit formula by taking the derivative of $f(x) = \ln x$ at $x = 1$:

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \ln(1 + x)^{1/x} = 1.$$

Applying the (continuous) exponential function to the limit we get

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

Note If we substitute $y = 1/x$ in the above limit we get

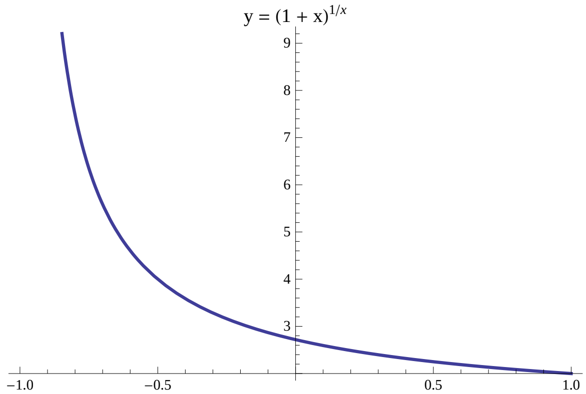
$$e = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y \quad \text{and} \quad e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

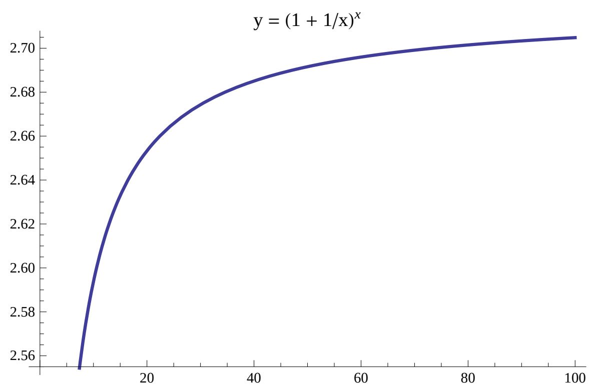
where n is an integer (see graphs below). We look at large values of n below to get an approximation of the value of e.

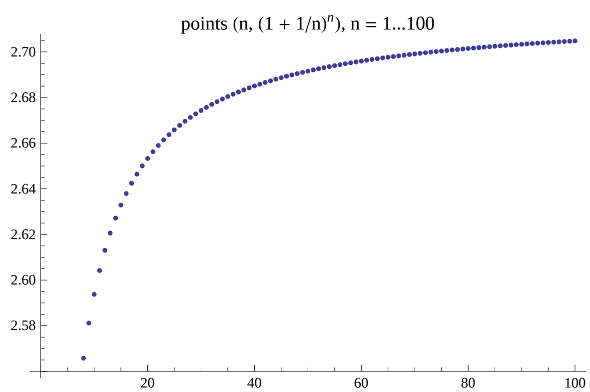
$n = 10 \to \left(1 + \frac{1}{n}\right)^n = 2.59374246, \quad n = 100 \to \left(1 + \frac{1}{n}\right)^n = 2.70481383,$

$n = 100 \to \left(1 + \frac{1}{n}\right)^n = 2.71692393, \quad n = 1000 \to \left(1 + \frac{1}{n}\right)^n = 2.1814593.$

Example Find $\lim_{x \to 0}(1 + \frac{x}{2})^{1/x}$.


y = (1 + x)^{1/x}


y = (1 + 1/x)^x


points (n, (1 + 1/n)^n), n = 1...100